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Delaware Estuary Comprehensive Study

Report Number 4

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Public Health Service

DELAWARE ESTUARY COMPREHENSIVE STUDY

TECHNICAL REPORT NO. 4

Mathematical Model for Dissolved Oxygen

This Technical Report, one of a series prepared by the staff of the Delaware Estuary Comprehensive Study is for the information of interested agencies and individuals. Reports will be added to the series as developments from the various phases of the Study provide additional information that will be of value to others concerned with water pollution control.

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No. 1 Report on Analysis of the Dye Diffusion Data in the Delaware River Estuary - Evaluation of Diffusion Coefficients

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Journal of the
SANITARY ENGINEERING DIVISION
Proceedings of the American Society of Civil Engineers

MATHEMATICAL MODEL FOR DISSOLVED OXYGEN

By Robert V. Thomann,[1] A. M. ASCE

SYNOPSIS

A mathematical model is presented for describing the time variation of dissolved oxygen in a finite number of sections in a body of water. A systems analysis technique is used, and the DO response equations resulting from the imposition of general input forcing functions are derived. An analytical relationship is developed between the time variation of sewage effluent in one section and the DO response in any other section. Although the use of systems techniques as given is completely general, the analytical model is presented for the case of DO variations in an estuary. Nontidal streams assume the role of a special case of the general formulation.

INTRODUCTION

A need exists for a general mathematical approach to the problem of describing the space and time variations of dissolved oxygen (DO) in a body of water. Also, the recent advent of continuous monitoring installations requires the use of new techniques for the adequate analysis of continuous or quasi-continuous water quality records. A mathematical model for DO based on the

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concepts of "systems analysis" is presented. This type of analysis permits the derivation of analytical relationships between treatment plant effluents and the resulting dissolved oxygen in the stream. This type of relationship is useful for many purposes, not the least of which is the assessment of the effects of treatment plant effluent loads.

In order to provide a manageable presentation, the verification of the model will not be given in detail. Hence, this paper presents the analytical techniques that may be used to solve many of the problems associated with describing the dissolved oxygen. The model was applied to a part of the Delaware estuary in order to verify an observed DO time series. More recently, the model was expanded to include the entire estuary and computational programs have been written to incorporate various levels of flow, diffusion, reaeration, and decay mechanisms. A discussion of the verification and numerical results will be the topic of separate presentations.

Previous work in this field, which uses a systems analysis approach, has been limited. R. Dorrestein[2,3] presented a model for the distribution of salinity in the Ems estuary. P. Welander[4] used some basic concepts of systems analysis to compute the response variation of carbon dioxide in the ocean to the variation of CO_2 in the atmosphere.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear and are arranged alphabetically in the Appendix.

DEFINITION OF A "SYSTEM"

A system may be defined as any mechanism that somehow transforms an input variable into an output variable. For example, a television set operates on an input signal—namely, spacial electromagnetic waves—and transforms this input into an output—the television picture. This transformation by the television set system is accomplished through a complex arrangement of electronic mechanisms. It is not necessary to understand the inherent nature of these mechanisms, rather a knowledge of the input together with a knowledge of the over-all characteristics of the set allows an explicit determination of the output.

The forces of nature and man that concern the sanitary engineer may be viewed in a similar manner. These forces are visualized as acting through a succession of systems each of which transforms an input into an output. For example, the saturation value of DO is known to be strongly dependent on water temperature. A record of temperature can then be considered as an input signal over time which acts as an input on a system. The output from this system is the DO saturation value. The system itself involves relatively complex molecular reactions and incorporates or "lumps" together many features of the dependence of the saturation variable on the temperature variable.

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2 "The Use of a Matrix-Function for Describing the Longitudinal Transfer of Matter in a Steady-State Estuary," by R. Dorrestein, Internatl. Oceanographic Congress Preprints, edited by M. Sears, AAAS, Washington, D. C., 1959, pp. 685-687.

3 "On the Mixing and Flushing of the Water in the Ems Estuary," by R. Dorrestein and L. Otto, Verhundel. Kon. Ned. Geol. Mijnb, K. Gen., Geol. Serie, D1 xix, pp. 84-100, 1960.

4 "On the Frequency Response of Some Different Models Describing the Transient Exchange of Matter Between the Atmosphere and the Sea," by P. Welander, Tellus, Vol. 11, No. 3, 1959, pp. 348 and 354.

As a second example, any body of water may be considered as a hydrodynamic system which accepts a number of input signals, such as temperature, waste treatment plant effluents, and plankton activity, combines and distributes these signals in certain ways throughout the hydraulic regime, and ultimately generates an output signal, for example, dissolved oxygen. When inspecting a series of DO analyses taken at a particular station in the field, the sanitary engineer is attempting to understand an output signal that represents the communication and combination of a number of forces in the environment. The record at any instant may be the result of actions that had occurred several weeks ago. The characteristics of the system indicate the particular combination of input signals both in the past and present which led to the final observed output value. The terms input, output, and system are thus defined in a broad sense.

The systems as treated herein have time as the independent variable although a system may also be treated in spacial coordinates or in angular displacements. A particular case of a general system is given by a linear system that may be mathematically defined (W. B. Davenport and W. L. Root;[5] D. K. Cheng[6]) as a system such that if an input $x_1(t)$ produces an output $y_1(t)$ then the input

$$x(t) = a_1 x_1 (t) + a_2 x_2 (t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

yields the output

$$y(t) = a_1 y_1 (t) + a_2 y_2 (t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Under this condition, the system obeys the principle of linear superposition. In other words, the output that results from a transformation of one input signal may be added algebraically to the output that results from a second input. If a_1 and a_2 are constants, the system is termed stationary. It should be noted that it is not necessary to restrict a_1 and a_2 to constant values, but in order to provide a tractable analysis at this point, they may be assumed as fixed parameters. The stationary condition implies that if the input is varying in time with a certain frequency (or period) of oscillation, then the output will respond with the same frequency of oscillation and new output frequencies will not be produced.

A SIMPLE LINEAR SYSTEM

With a stationary linear system defined, the following well-known simple equation may be considered:

$$\frac{dc}{dt} = r \left[c_s(t) - c(t)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

5 "An Introduction to the Theory of Random Signals and Noise," by W. B. Davenport and W. L. Root, McGraw-Hill Book Co., Inc., New York, N. Y., 1958, pp. ix and 1-393.

6 "Analysis of Linear Systems," by D. K. Cheng, Addison-Wesley Publishing Co., Reading, Mass., 1959, pp. xiii and 1-431.

or

$$\frac{dc}{dt} + rc(t) = r\,c_s(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which c(t) represents DO, in milligrams per liter; $c_S(t)$ represents the saturation value of DO; and r is the reaeration coefficient.

The saturation value is a function of the water temperature. Because over a one year period it is reasonable to consider that water temperature varies as a periodic function, the saturation value will also vary in a periodic fashion. The $c_S(t)$ is given by a harmonic function of the form

$$c_s(t) = C_{os} \cos(\omega t + \theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which C_{os} represents the amplitude of c_S; ω = frequency = $2\pi/T$; T represents the fundamental period of oscillation (365 days); and θ is the phase angle measured from January 1 = 0°. Eq. 5 can also be written in complex notation using the well-known formula

$$\cos\alpha = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

In this notation, Eq. 5 is

$$c_s(t) = \frac{C_{os}}{2} e^{i(\omega t+\theta)} + e^{-i(\omega t+\theta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Eq. 5 can then be considered as the sum of two functions. Because Eq. 3 is a linear equation, the solutions can be linearly superimposed. If the positive part of $c_S(t)$ is taken first, i.e.,

$$c_{sp} = \frac{C_{os}}{2} e^{i(\omega t+\theta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

then the solution to Eq. 3, if Eq. 8 is given, is obtained by heuristic reasoning and can be written as

$$c_p = \frac{C_{os}}{2} \phi(i\omega)\, e^{i(\omega t+\theta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

in which $\phi(i\omega)$ must be determined. This complex function of the frequency is termed the transfer function of the system. If Eqs. 8 and 9 are substituted into Eq. 3, $\phi(i\omega)$ is given by

$$\phi(i\omega) = \frac{r}{r+i\omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Any complex function such as $\phi(i\omega)$ can also be written in terms of real and imaginary parts or its modulus and phase angle. Hence,

$$\phi(i\omega) = P + iQ = A\, e^{i\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

in which $P(\omega)$ represents the real part of $\phi(i\omega)$; $Q(\omega)$ represents the imaginary part of $\phi(i\omega)$; $A(\omega)$ is the modulus of $\phi(i\omega) = \sqrt{P^2 + Q^2}$; $\alpha(\omega)$ is the phase shift = arctg (Q/P); and

$$c_p(t) = \frac{C_{os}}{2} A(\omega)\, e^{i[\omega t + \theta + \alpha(\omega)]} \quad \ldots\ldots\ldots\ldots (12)$$

If now the second part of $c_s(t)$ is substituted, i.e.,

$$c_{sn}(t) = \frac{C_{os}}{2} e^{-i(\omega t + \theta)} \quad \ldots\ldots\ldots\ldots\ldots (13)$$

and a similar procedure is followed, the second solution to $c(t)$ is obtained as

$$c_n(t) = \frac{C_{os}}{2} \phi(-i\omega)\, e^{-i(\omega t + \theta)} \quad \ldots\ldots\ldots\ldots\ldots (14)$$

in which

$$\phi(-i\omega) = \frac{r}{r - i\omega} = A(\omega)\, e^{-i\alpha(\omega)} \quad \ldots\ldots\ldots\ldots\ldots (15)$$

The total solution is then given by

$$c(t) = c_p + c_n = \frac{C_{os}}{2} A(\omega) \left\{ e^{i[\omega t + \theta + \alpha(\omega)]} + e^{-i[\omega t + \theta + \alpha(\omega)]} \right\} \quad \ldots (16)$$

or

$$c(t) = C_{os}\, A(\omega) \cos [\omega t + \theta + \alpha(\omega)] \quad \ldots\ldots\ldots\ldots (17)$$

Eq. 17 indicates that the knowledge of $\phi(i\omega)$ [hence $A(\omega)$ and $\alpha(\omega)$] provides complete information on the output response [$c(t)$] to the varying input signal $c_s(t)$. The amplitude C_{os} of the input is attenuated by $A(\omega)$, whereas the shift in phase is given by $\alpha(\omega)$. The real and imaginary parts, P and Q of $\phi(i\omega)$, are obtained from Eq. 10 and are given by

$$P(\omega) = \frac{r^2}{r^2 + \omega^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

and

$$Q(\omega) = \frac{r\omega}{r^2 + \omega^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

and the attenuation and phase shift are given by

$$A(\omega) = \frac{r}{\sqrt{r^2 + \omega^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

and

$$\alpha(\omega) = \text{arctg}\left(\frac{\omega}{r}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

It is seen that the transfer function does not depend on the input but reflects the physical properties of the system; in this case, the reaeration coefficient. A block diagram of the linear system described by Eq. 3 is shown in Fig. 1.

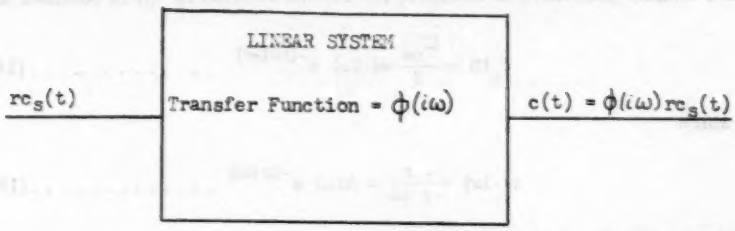

FIG. 1.—BLOCK DIAGRAM REPRESENTATION OF A SIMPLE LINEAR SYSTEM

GENERAL CASE

The preceding example is certainly oversimplified and is rarely met in nature. Several other forces act in a body of water to produce changes in the dissolved oxygen. Furthermore, the foregoing system exists as a separate physical entity with no relationships to similar systems in its neighborhood. In most physical situations, the systems are coupled together in several different ways. For a general body of water, the DO of one system is coupled to the DO of other systems by the mechanisms of advection (or volume transport of DO) and diffusive exchange. In the most general case, all of the systems are mutually interconnected with each other. Fig. 2 illustrates schematically a hypothetical body of water that has been segmented into four sections, all of which are interconnected. In the following general development,

there is no preferred direction of flow. This allows flexibility in the application of the model to various specific conditions.

The first assumption then in the development of a general model is that a body of water (lake, stream, estuary, etc.) can be segmented into a discrete number of parts. In the actual case, the medium is of course continuous, but a finite number of sections is believed to be valid because descriptions of quality in the space domain are rarely desired on a continuous basis. In other words, descriptions of, for example, the longitudinal distribution of DO are

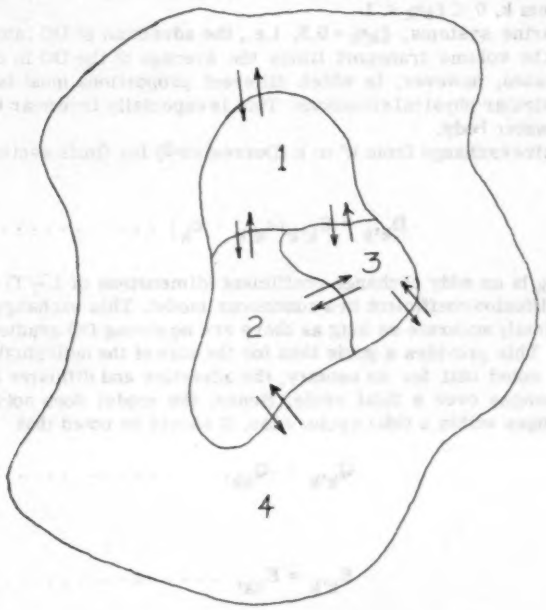

FIG. 2.—SCHEMATIC REPRESENTATION OF A FOUR-SECTION BODY OF WATER

not necessary for every foot of a stream. Significant DO gradients usually do not exist on so small a scale. The determination of how the particular body of water is to be segmented depends on the specific situation and some general guidelines will be given as the derivation continues.

The time rate of change of DO in any of the sections, each of which is a linear system depends on (a) the advection and diffusion of DO both into and out of the system and (b) the sources and sinks of DO within each section. These additions or removals of DO are considered as the variables that "force" the DO to respond. They are termed the forcing functions of the system. In the preceding example, the varying saturation value assumed the role of an input forcing function.

If then n linear systems are considered, all mutually coupled and if the kth system is examined, the advection into k from neighboring systems k' may be given by

$$A_{k'k} = Q_{k'k}\left[\xi_{k'k}\, c_{k'} + (1 - \xi_{k'k})c_k\right] \quad \ldots\ldots\ldots (22)$$

in which $Q_{k'k}$ represents the volume transport of water from k' to k, c_k is the DO concentration in k, $c_{k'}$ is the DO concentration in k', and $\xi_{k'k}$ is a factor representing that fraction of $c_{k'}$ that is advected together with $(1 - \xi_{k'k})c_k$ into the system k, $0 < \xi_{k'k} < 1$.

For estuarine systems, $\xi_{k'k} = 0.5$, i.e., the advection of DO into system k consists of the volume transport times the average of the DO in c_k and $c_{k'}$. There are cases, however, in which different proportions must be advected based on particular physical situations. This is especially true near the boundaries of the water body.

The diffusive exchange from k' to k (Dorrestein[2]) for finite sections can be written as

$$D_{k'k} = E_{k'k}(c_{k'k} - c_k) \quad \ldots\ldots\ldots\ldots\ldots (23)$$

in which $E_{k'k}$ is an eddy exchange coefficient (dimensions of L^3/T) analogous to the eddy diffusion coefficient in a continuous model. This exchange assumption is sufficiently accurate as long as there are no strong DO gradients within each section. This provides a guide then for the size of the individual systems.

It can be noted that for an estuary, the advective and diffusive exchanges are net exchanges over a tidal cycle. Hence, the model does not attempt to describe changes within a tidal cycle. Also, it should be noted that

$$Q_{k'k} = -Q_{kk'} \quad \ldots\ldots\ldots\ldots\ldots (24)$$

and

$$E_{k'k} = E_{kk'} \quad \ldots\ldots\ldots\ldots\ldots (25)$$

The primary sources and sinks of DO are: (a) Atmospheric reaeration (the simple model previously given), (b) removal of DO due to the presence of organic matter (BOD), and (c) the respiratory and photosynthetic action of aquatic plants.

Production or utilization of oxygen due to chemical reactions and all remaining sources and sinks are not considered in an analytical manner, although these forcing functions do not present any conceptual difficulties as will be seen subsequently.

In order to compute the total rate of change of DO in system k, the diffusion and advection from all systems bordering on k must be included. This leads to the following DO equation for the k^{th} system:

$$V_k \frac{dc_k}{dt} = \sum_{k'} \left\{ Q_{k'k}\left[\xi_{k'k} c_{k'} + (1 - \xi_{k'k}) c_k\right] + E_{k'k}(c_{k'} - c_k) \right\}$$

$$+ V_k r_k \left[c_{sk}(t) - c_k\right] - V_k d_k(t) L_k(t) + S_k(t) + M_k(t) \quad \ldots\ldots (26)$$

in which V_k is the volume of the section; $d_k L_k$ represents the sink of DO due to ultimate BOD, L_k, and decay rate, d_k; S_k represents DO from phytoplankton in V_k; and M_k represents all other sources and sinks of DO in V_k.

If $k = 1$ in Eq. 26 and the various terms are expanded and collected, the result is

$$\frac{dc_1}{dt} + \left[r_1 - (1 - \xi_{21}) \frac{Q_{21}}{V_1} - \ldots - (1 - \xi_{n1}) \frac{Q_{n1}}{V_1} + \frac{E_{21}}{V_1} + \ldots + \frac{E_{n1}}{V_1} \right] c_1$$

$$- \left(\xi_{21} \frac{Q_{21}}{V_1} + \frac{E_{21}}{V_1} \right) c_2 - \ldots - \left(\xi_{n1} \frac{Q_{n1}}{V_1} + \frac{E_{n1}}{V_1} \right) c_n$$

$$= r_1 c_{s1} - d_1 L_1 + S_1 + M_1 \quad \ldots\ldots\ldots\ldots\ldots (27)$$

If

$$a_{1j} = \frac{1}{V_1} \left(\xi_{1j} Q_{1j} - E_{1j} \right) \quad (j = 2, 3, \ldots n) \quad \ldots\ldots\ldots (28)$$

and

$$a_{11}(p) = p + R_{11} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

in which

$$p \equiv \frac{d}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

and

$$R_{11} = r_1 + \frac{1}{V_1} \left\{ \sum_{j=2}^{n} \left[(1 - \xi_{1j}) Q_{1j} + E_{1j} \right] \right\} \quad \ldots\ldots\ldots (31)$$

then Eq. 27 is

$$a_{11}(p)\, c_1 + a_{12}\, c_2 + a_{13}\, c_3 + \ldots + a_{1n}\, c_n = f_1(t) \quad \ldots\ldots (32)$$

in which

$$f_1(t) = r_1\, c_{s1} - d_1 L_1 + S_1 + M_1 \quad \ldots\ldots\ldots\ldots (33)$$

Eq. 32 represents a stationary linear system for $c_1(t)$ with the sources and sinks of DO as forcing functions and also with feed-back mechanisms from adjoining systems. A similar procedure can be followed for $k = 2,3 \ldots, n$ and the individual system equations for $c_k(t)$ can then be obtained. For all k values then, the entire system is given by

$$\left.\begin{aligned}
a_{11}(p)c_1 + a_{12}c_2 + a_{13}c_3 + \ldots + a_{1n}c_n &= f_1(t)\\
a_{21}c_1 + a_{22}(p)c_2 + a_{23}c_3 + \ldots + a_{2n}c_n &= f_2(t)\\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot &\\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot &\\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot &\\
a_{n1}c_1 + a_{n2}c_2 + a_{n3}c_3 + \ldots + a_{nn}(p)c_n &= f_n(t)
\end{aligned}\right\} \quad \ldots (34)$$

Eq. 34 represents the complete linear system with constant coefficients off the diagonal, first order differential operators plus constants on the diagonal and all forcing functions on the right side. It should be recalled that the constants in the a_{ij} terms embody the hydrodynamic characteristics of the body of water, namely, the net amount of flow, the magnitude of the turbulent exchange and the reaeration properties. Because the n sub-systems that form the over-all system are all linear, the various solutions that may be obtained can be linearly superimposed. This means that all forcing functions may be assumed to be zero except one and the system solved for any $c_j(t)$. A second forcing function with all others equal to zero can then be imposed and a second solution for c_j is determined. In this manner, the response of $c_j(t)$ to each $f_i(t)$ can be obtained separately, and the individual solutions can then be superimposed.

If then $f_2(t), f_3(t) \ldots f_n(t) = 0$ and a solution for $c_1(t)$ is desired, the process of successive elimination yields an n^{th} order differential equation (Solodovnikov[7]):

$$a_n \frac{d^n c_1}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} c_1 + \ldots + a_0 c_1 = b_m \frac{d^m f_1}{dt^m} + \ldots + b_0 f_1 \quad \ldots\ldots (35)$$

7 "Introduction to the Statistical Dynamics of Automatic Control Systems," by V. V. Solodovnikov, Dover Publishing Co., Inc., New York, N. Y., 1960, pp. xx and 1-307.

in which a_k may be determined from the determinant D(p):

$$D(p) = a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_0 \quad \ldots\ldots\ldots (36)$$

in which

$$D(p) = \begin{vmatrix} a_{11}(p) & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}(p) & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}(p) \end{vmatrix} \quad \ldots\ldots\ldots (37)$$

and b_k may be determined from

$$M_{ij}(p) = b_{m,ij} p^m + b_{m-1,ij} p^{m-1} + \ldots b_{0,ij} \quad \ldots\ldots\ldots (38)$$

in which $M_{ij}(p)$ is the signed minor of D(p) that results from cancelling the i^{th} row, the one in which the forcing function is applied, and the j^{th} column, the one in which the desired $c_j(t)$ appears. Hence, for $f_1(t)$ and $c_1(t)$, $M_{11}(p)$ is

$$\begin{vmatrix} a_{22}(p) & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{32} & a_{33}(p) & a_{34} & \cdots & a_{3n} \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn}(p) \end{vmatrix}$$

It is seen that $m < n$. Eq. 35 may also be written as

$$D(p)\ c_1 = M_{11}(p)\ f_1(t) \quad \ldots\ldots\ldots\ldots\ldots (39)$$

and the complete set of equations for $c_1(t)$ that must be solved and whose solutions would be linearly superimposed is given by

$$\left.\begin{aligned} D(p)\ c_1 &= M_{11}(p)\ f_1 \\ D(p)\ c_1 &= M_{21}(p)\ f_2 \\ &\cdot \qquad\quad \cdot \\ &\cdot \qquad\quad \cdot \\ &\cdot \qquad\quad \cdot \\ D(p)\ c_1 &= M_{n1}(p)\ f_n \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (40)$$

In general, the equation for any $c_j(t)$ with the forcing function applied in system i is

$$D(p)\ c_j = M_{ij}(p)\ f_i(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

Now $f_i(t)$ in Eq. 41 is a sum of several forcing functions, primarily the reaeration from the atmosphere, removal of DO due to the BOD, and photosynthesis. Because the systems are all linear, each one of these separate forcing functions may be taken independently in Eq. 41. Moreover, each separate part of $f_i(t)$ may itself be composed of other functions. This was indicated in the simple reaeration example in which $c_s(t)$ was considered as the sum of two exponential forcing functions. Hence, $f_i(t)$ may be considered in a similar manner, i.e.,

$$f_i(t) = \frac{f_0}{2}\, e^{i\omega t} + \frac{f_0}{2}\, e^{-i\omega t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

in which f_0 is the input amplitude and the input phase angle is assumed to be zero. If the positive exponential is taken first, then the solution to Eq. 41 is given by

$$c_{jp} = \frac{f_0}{2}\, \phi_{ij}(i\omega)\ e^{i\omega t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (43)$$

in which $\phi(i\omega)$, $i=\sqrt{-1}$, is the desired transfer function that characterizes the DO response to $f_i(t)$. This function can be obtained by substituting the positive exponential of Eqs. 42 and 43 into Eq. 41 and simplifying the result. The transfer function is then given by

$$\phi_{ij}(i\omega) = \frac{b_{m_{ij}}(i\omega)^m + \ldots + b_{1_{ij}}(i\omega) + b_{0_{ij}}}{a_n(i\omega)^n + \ldots + a_1(i\omega) + a_0} \quad \ldots\ldots\ldots\ldots (44)$$

and the real and imaginary parts are

$$\phi_{ij}(i\omega) = \frac{\alpha_{ij}(\omega) + i\beta_{ij}(\omega)}{\gamma(\omega) + i\delta(\omega)} \quad \ldots\ldots\ldots\ldots\ldots (45)$$

in which

$$\alpha_{ij}(\omega) = b_{0_{ij}} - b_{2_{ij}}\omega^2 + b_{4_{ij}}\omega^4 - \ldots \quad \ldots\ldots\ldots (46)$$

$$\beta_{ij}(\omega) = b_{1_{ij}}\omega - b_{3_{ij}}\omega^3 + b_{5_{ij}}\omega^5 - \ldots \quad \ldots\ldots\ldots (47)$$

$$\gamma(\omega) = a_0 - a_2\omega^2 + a_4\omega^4 - \ldots \quad \ldots\ldots\ldots\ldots (48)$$

and

$$\delta(\omega) = a_1\omega - a_3\omega^3 + a_5\omega^5 - \ldots \quad \ldots\ldots\ldots\ldots (49)$$

As with the simple reaeration model, Eq. 45 may also be written in the following forms:

$$\phi_{ij}(i\omega) = P_{ij}(\omega) + iQ_{ij}(\omega) = A_{ij}(\omega)\, e^{i\theta_{ij}(\omega)} \quad \ldots\ldots\ldots (50)$$

in which

$$P = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

$$A = \sqrt{P^2 + Q^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (52)$$

$$Q = \frac{\beta\gamma - \alpha\delta}{\gamma^2 - \delta^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (53)$$

and

$$\theta = \text{arctg}\left(\frac{Q}{P}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

A parallel analysis for the negative part of the input yields similar results. The total solution is then obtained as a linear superposition of the two separate solutions and is given by

$$c_j(t) \frac{f_{0i}}{2} \left[\phi_{ij}(i\omega)\ e^{i\omega t} + \phi_{ij}(-i\omega)\ e^{-i\omega t} \right]$$

$$= \frac{f_{0i}}{2} A_{ij}(\omega) \left[e^{i(\omega t+\theta)} + e^{-i(\omega t+\theta)} \right] \quad \ldots\ldots\ldots\ldots (55)$$

$$c_j(t) = f_{0i} A_{ij}(\omega)\ \cos\left[\omega t + \theta_{ij}(\omega) \right] \quad \ldots\ldots\ldots\ldots (56)$$

Eq. 56 indicates that the DO response due to a harmonic input, $f_i(t)$ is given completely by knowledge of $A_{ij}(\omega)$, the amplitude attenuation, and the phase shift, $\theta_{ij}(\omega)$. It may be noted that these functions are characteristics of the system and do not depend on the nature of the input. For the DO model given herein, the transfer function (given by $A_{ij}(\omega)$ and $\theta_{ij}(\omega)$) is a function of frequency and also of the reaeration coefficient and the diffusive and advective exchange between sections. Hence, the transfer function indicates how the hydrodynamic environment acts on an input forcing function to produce the DO output.

The model also provides an analytical tool for assessing the effects of an input variable in one section on the output DO in any other section. Therefore, if there is a particular significance attached to, for example $f_5(t)$, which may be the BOD in section 5, the DO response in any other section, for example section 8, can be obtained explicitly through the transfer function, $\phi_{58}(i\omega)$.

RESPONSE TO GENERAL FORCING FUNCTION

The foregoing analysis provides the necessary DO response information to a harmonic input. This response information is embodied in the transfer function. However, the model is not limited to periodic inputs and the DO response due to a general time varying forcing function can be computed. If the input is composed of several harmonics, then of course each harmonic can be imposed separately and the total response is then given by a superposition of the individual responses.

If, however, the input is a transient phenomenon, the analysis proceeds in a different manner. One type of transient that can be considered is an input for which a Fourier transformation exists, i.e., $f_i(t)$ is of the form

$$f_i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_i(i\omega)\ e^{i\omega t}\ d\omega \quad \ldots\ldots\ldots\ldots (57)$$

in which

$$F_i(i\omega) = \int_{-\infty}^{\infty} f_i(t)\ e^{-i\omega t}\ dt \quad \ldots\ldots\ldots\ldots (58)$$

Eqs. 57 and 58 constitute a Fourier transform pair and permit transformation from the time domain to the frequency domain. With an input given by Eq. 57, it can be shown (Solodovnikov[7]) that the output is obtained by the following equation:

$$\tilde{c}_j(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{ij}(i\omega)\ F_i(i\omega)\ e^{i\omega t}\ d\omega \quad \ldots\ldots\ldots (59)$$

Eq. 59 indicates that the transient output in section j, $\tilde{c}_j(t)$, is obtained from the Fourier transform of the input in section i and the transfer function between sections i and j. In some cases, however, the transformation from the time domain to the frequency domain via Eqs. 57 and 58 is not advantageous. This occurs when the transient input, $f_i(t)$, is of such a nature as to produce a large number of frequencies after the transformation is made. This can be illustrated most readily by considering the input to be a sharp spike at time $t = t_0$. Mathematically, a spike in the time domain can be represented by a so-called Dirac delta function $\delta(t-t_0)$. This function is defined to be zero for all values of t except at $t = t_0$ where it tends to infinity while the following relationship applies:

$$\int_{-\infty}^{\infty} \delta(t-t_0)\ dt = 1 \quad \ldots\ldots\ldots\ldots\ldots (60)$$

or

$$m \int_{-\infty}^{\infty} \delta(t-t_0)\ dt = m \quad \ldots\ldots\ldots\ldots\ldots (61)$$

in which m is a constant. If

$$f_i(t) = m\ \delta(t-t_0) \quad \ldots\ldots\ldots\ldots\ldots (62)$$

then the Fourier transformation as given by Eq. 57 is

$$F_i(i\omega) = m \int_{-\infty}^{\infty} \delta(t-t_0)\ e^{-i\omega t}\ dt \quad \ldots\ldots\ldots\ldots (63)$$

It is readily seen then that through the use of Eq. 61, the Fourier transform is

$$F_i(i\omega) = m \quad \ldots\ldots\ldots\ldots\ldots (64)$$

Eq. 64 shows that a spike in time space transforms to an infinite number of components in the frequency space. Hence, a large number of frequency components would have to be used as inputs into the linear system and all solutions superimposed to determine the DO response to a sharp transient in the time domain. Clearly, this leads to an extremely cumbersome analysis, and it would be preferable to remain in the time domain entirely and to obtain a function analogous to the transfer function in the frequency domain.

This analogous function is termed the impulsive response or weighting function and represents the response of the system to an input delta function. It can be obtained from Eq. 59 where, as was shown previously, the Fourier transform of a delta function is unity. Hence,

$$k_{ij}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{ij}(i\omega)\ e^{i\omega t}\ d\omega \quad \ldots\ldots\ldots\ldots (65)$$

in which $k_{ij}(t)$ is the impulsive response function. Eq. 65 shows that this function is a transformation of the transfer function and is also a characteristic of the hydrodynamic properties of a body of water. (It should be recalled that $\phi = \phi(E,Q,r)$, that is, the turbulent exchange, advective exchange, and reaeration coefficient, respectively.) Eq. 65 provides the means for conducting the analysis either in time or frequency spaces.

The final step is to determine the DO response to any type of input if the impulsive response function is known. This is accomplished by first approximating the shape of the input by a series of delta functions and then passing to the limit as the time interval between the functions goes to zero. The final result can be written as

$$\tilde{c}_j(t) = \int_0^{\infty} f(t-\tau)\ k\ (\tau)\ d\tau \quad \ldots\ldots\ldots\ldots\ldots (66)$$

It is seen from Eq. 66 that the impulsive response functions weigh the previous values of f(t) which results in the present value of $\tilde{c}(t)$. Hence, the longer k(t) remains significant, the more past values of f(t) influences the present value of the DO. This is particularly useful because it indicates how long a sharp transient input remains in a body of water and hence affects the oxygen content.

There is one final type of input that has not been described. This is an input that is a random or nearly random time series that cannot be characterized adequately by a summation of a large number of harmonics. The probability distribution of a time series of this nature often can be considered as a Gaussian or normal distribution. There are several such important functions that are inputs in the dissolved oxygen system. Notable among these functions is the BOD discharge from sewage treatment plants that has been treated as a random process (Thomann[8]). A full presentation of how random time series can be characterized and how the random DO response can be computed is beyond the scope of this report. It is important to note, however, that the DO response is again given completely by a knowledge of the transfer function and the random input.

In summary, the first input forcing function was assumed to be harmonic. It was then indicated that if the input was composed of several distinct harmonics, each harmonic could be imposed on the system and independent solutions could be obtained. The total response to all of the harmonics is then given by a superposition of the individual solutions.

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[8] "The Use of Systems Analysis to Describe the Time Variation of Dissolved Oxygen in a Tidal Stream," by R. V. Thomann, thesis presented to the Department of Meteorology and Oceanography, New York University of New York, N. Y., in 19--, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

The response to an input transient was then given by the transfer function and the Fourier transform of the input. However, in some cases, there is little advantage to transforming to frequency space because of the large number of harmonics that may result. Hence, the impulsive response function analogous to the transfer function was derived as the response of the system to an input spike.

Finally, any type of input can be imposed on the system and the response is then given by Eq. 66. The flexibility of a linear systems analysis is now evident. Those input phenomena that are largely frequency dependent (e.g., the saturation value) can assume the role of a forcing function first and the gross features of the DO response can then be described.

For sharp input transients (e.g., storm water overflows) the analysis shifts to the time domain and the transient response can be computed. The total DO response is then given by the sum of the harmonic response and the transient response.

Also, the BOD input forcing function in one sub-system and the DO response in any other sub-system can then be computed. In this manner, the ability exists to assess the effects of BOD loads on the DO regime independently of presence of other forcing functions, e.g., photosynthesis or the variation of the saturation value. This assumes of course that the BOD reaction proceeds independently of the DO concentration at any time.

AN IMPORTANT SPECIAL CASE: THE ONE-DIMENSIONAL ESTUARY

For the one-dimensional case, the individual systems are strung longitudinally down the length of the estuary. It may be assumed that the diffusive-advective phenomena are net exchanges over a tidal cycle. Hence, the flow proceeds in one direction only, i.e., there is no net flow upstream.

The entire system for the one-dimensional case is given by

$$\left.\begin{aligned}
a_{11}(p)c_1 + a_{12}c_2 + 0 + \ldots + 0 &= f_1(t)\\
a_{21}c_1 + a_{22}(p)c_2 + a_{23}c_3 + 0 + \ldots + 0 &= f_2(t)\\
0 + a_{32}c_2 + a_{33}(p)c_3 + a_{34}c_4 + 0 + \ldots + 0 &= f_3(t)\\
\cdot \quad \cdot \quad \cdot &\\
\cdot \quad \cdot \quad \cdot &\\
\cdot \quad \cdot \quad \cdot &\\
0 + \ldots + 0 + a_{n,n-1}c_{n-1} + a_{nn}(p)c_n &= f_n(t)
\end{aligned}\right\} \quad \ldots (67)$$

It is seen that the system of equations is tridiagonal which somewhat simplifies the expansion of the determinant and its minors. If the flow and exchange coefficient are constant from section to section and if the advection factor is

set equal to 0.5, then the elements in Eq. 67 are given by

$$R_{ii} = r_i + \frac{2E}{V_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots (68)$$

$$a_{ij} = \frac{1}{V_i} (0.5\, Q - E) \quad \ldots\ldots\ldots\ldots\ldots\ldots (69)$$

and

$$a_{ji} = -\frac{1}{v_i} (0.5\, Q + E) \quad \ldots\ldots\ldots\ldots\ldots\ldots (70)$$

These assumptions are made here only for convenience and are not necessary, but as will be seen, they lead to some useful general statements.

It may be noted from Eqs. 45 and 50 that the real part of the transfer function for $\omega \to o$ is given approximately by

$$P_{ij}\ (\omega \to o) = \frac{b_{0_{ij}}}{a_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (71)$$

and that the imaginary part, $Q_{ij}(\omega)$, goes approximately to zero. For small ω then, an examination of b_0 and a_0 will provide a means for determining the physical characteristics of the transfer function for a one-dimensional estuary. The investigation of P_{ij} for small ω is pertinent because the largest part of the variation of input forcing functions and DO response is due to long period (small frequency) phenomena.

Now, the b_{0ij} coefficients are given by

$$b_{0j-1,j} = -a_{ji} \left[R^{n-2} - \alpha_{j-1,j} \left(a_{ij}a_{ji}\right) R^{n-4} + \beta_{j-1,j} \left(a_{ij}a_{ji}\right)^2 R^{n-6} - \ldots \right] \quad \ldots\ldots\ldots\ldots (72a)$$

$$b_{0j,j} = R^{n-1} - \alpha_{jj} \left(a_{ij}a_{ji}\right) R^{n-3} + \beta_{jj} \left(a_{ij}a_{ji}\right)^2 R^{n-5} - \ldots\ldots (72b)$$

$$b_{0j+1,j} = -a_{ij}\, b_{0j-1,j} \quad \ldots\ldots\ldots\ldots\ldots\ldots (72c)$$

in which α_{ij}, β_{ij} . . . are constants. The a_0 coefficient is given by

$$a_0 = R\, b_{0jj} + 2\, a_{ij}\, a_{ji}\, b_{0j-1,j} \quad \ldots\ldots\ldots\ldots\ldots\ldots (73)$$

If in Eqs. 72 and 73 the cross-product terms representing corrections to a diagonal determinant are neglected, the coefficients are

$$b_{0j-1,j} = -a_{ji}\, R^{n-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (74)$$

$$b_{0jj} = R^{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (75)$$

$$b_{0j+1,j} = -a_{ij}\, R^{n-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (76)$$

and

$$a_o = R^n \quad \ldots\ldots\ldots\ldots\ldots\ldots (77)$$

so that

$$P_{j-1,j}(\omega \to 0) = \frac{(0.5\ Q + E)}{V\left(r + \frac{2E}{V}\right)^2} \quad \ldots\ldots\ldots\ldots (78a)$$

$$P_{j,j}(\omega \to 0) = \frac{1}{\left(r + \frac{2E}{V}\right)} \quad \ldots\ldots\ldots\ldots (78b)$$

and

$$P_{j+1,j}(\omega \to 0) = \frac{(E - 0.5\ Q)}{V\left(r + \frac{2E}{V}\right)^2} \quad \ldots\ldots\ldots\ldots (78c)$$

For forcing functions in systems j-k, (k=1,2,. . .j-1),

$$P_{j-k,j}(\omega \to 0) = \frac{(0.5\ Q + E)^k}{V^k\left(r + \frac{2E}{V}\right)^{k+1}} \quad \ldots\ldots\ldots\ldots (79)$$

and for forcing functions in systems j+q, (q=1,2,. . .n-j),

$$P_{j+q,j}(\omega \to 0) = \frac{(E - 0.5\ Q)^q}{V^q\left(r + \frac{2E}{V}\right)^{q+1}} \quad \ldots\ldots\ldots\ldots (80)$$

If $P_{j,j}(\omega)$ is considered first, it is seen that the DO response is essentially independent of the advective effect. (This effect is present but occurs in the cross-product correction terms which are relatively small.) Also, for negli-

gible turbulent exchange, the response is the simple gradient exchange model shown in Fig. 1. Increased turbulent exchange decreases the DO response. This is to be expected because increased turbulence would tend to destroy DO gradients and oxygen would be diffused rapidly into segment j. The DO response in j would then be small and would tend to remain at mean steady state conditions. It is important to note the effect of changing the volume of the segments. For large V, the response again approaches the gradient model and for small V the turbulent exchange is increased, which results in a decreased response.

When $P_{j-k,j}(\omega)$ is examined, it is noted that the DO response is directly proportional to $0.5Q^k$. Hence, for increasing flows, forcing functions in systems further removed from system j must be considered. This means that as Q is increased, attention must be focused on the more distant organic sinks which may be due, for example, to discharges from sewage treatment plants. The analysis then of this particular one-dimensional model pointedly illustrates the effect of increasing the fresh water inflow at the head of the estuary. Indeed this has been proposed for a number of estuaries and has led to subsequent controversies over the effect on DO due to increased flow from upstream impoundments.

Changes in the diffusive effect for $P_{j-k,j}(\omega)$ have the same result as for $P_{i,j}(\omega)$. However, as the volume of the segments is increased, the response goes to zero. As the volume is decreased the response increases and in the limit becomes indeterminant. Hence, the volumes of the segments should not be so large as to result in the simple gradient model nor so small as to have responses in distant systems increase without limit. Both conditions would be physically undesirable. Of course, the primary criterion is that the volume should not be so large as to include a significant DO gradient in the length of the segment. This would violate the exchange mechanism postulated earlier.

For $P_{j+q,j}(\omega)$, it is seen that the increasing flow decreases the response, which is to be expected. The effect of diffusion is the same as described previously. However, a point may be reached at which $E = 0.5Q$, making the DO response equal to zero when forcing functions in downstream systems are applied. Under these conditions, the diffusive effect just balances the advective effect, and downstream sources and sinks assume no significance in the upstream DO response.

STABILITY CONSIDERATIONS

The final analysis that must be made on the linear system as given by Eq. 35 is concerned with the question of stability. Stability may be defined as that property of a system that tends to dampen disturbances during the initial transient stages of a process prior to the maintenance of the steady state conditions. Hence, the complementary solution to Eq. 35 must go to zero as time goes to infinity. Obviously, the term stability is relative and the degree of stability depends on the particular parameters of the system under investigation. The following discussion and the presentation of a criterion for determining stability are described in greater detail by B. M. Brown.[9]

[9] "The Mathematical Theory of Linear Systems," by B. M. Brown, John Wiley & Sons, Inc., New York, N. Y., 1961, pp. xv and 1-267, and Appendix 10.

The transient stage of the response of the linear system is given by Eq. 35 with the right side equal to zero, i.e.,

$$a_n \frac{d^n c}{dt^n} + a_{n-1} \frac{d^{n-1} c}{dt^{n-1}} + \ldots + a_1 \frac{dc}{dt} + a_0 c = 0 \quad \ldots\ldots\ldots (81)$$

The solution of Eq. 81 is of the form

$$\sum_{r=1}^{n} C_r e^{\lambda_r t}$$

in which $\lambda_1, \lambda_2 \ldots \lambda_n$ are the roots of the characteristic equation

$$F(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0 \quad \ldots\ldots\ldots (82)$$

or, reversing the order of the coefficients

$$F(\lambda) = k_o \lambda^n + k_1 \lambda^{n-1} + \ldots k_{n-1} \lambda + k_n = 0 \ldots\ldots\ldots (83)$$

It can readily be seen that regardless of the initial conditions, the system tends to stability only if the real parts of all the λ's are negative. It is also noted that this is independent of the forcing functions. Hence, if the zeros of $F(\lambda)$ are plotted in the complex plane, the necessary and sufficient condition for stability is that all the zeros must lie in the left half-plane. If the system is thus stable, the most significant zeros are those located nearest the imaginary axis because these will result in solutions that diminish most slowly. When all the zeros lie in the left half-plane, there are three possible results. If λ_1 is a negative real zero, which leads to a term of the form $e^{\lambda_1 t}$, the oscillation diminishes aperiodically. A pair of conjugate complex zeros such as λ_2 and λ_3 leads to terms of the type e^{at} cos bt and e^{at} sinbt in which $\lambda_2, \lambda_3 = a \pm bi$; these terms represent a damped oscillation. Finally, when λ_4 and λ_5 are conjugate imaginary zeros of the form $\pm ib$, the terms lead to an undamped oscillation of frequency b and the system is then said to be critically stable.

Of course, before an analysis of the zeros can be made, $F(\lambda)$ must be solved. This presents difficulties for higher order equations and numerical techniques must be used in order to determine the roots. Because of this difficulty, several techniques have been devised that provide the necessary information on stability without determining the zeros explicitly. The criterion of Routh is used here because of the ease of computation. Only the general rule is given; formal proof of the criterion can be found in the work by Brown[9] (Appendix 10).

According to Routh's criterion, the coefficients are written in two alternate rows and a third row is then formed from the preceding two. For an equation

of the form of Eq. 83, the rows are constructed as follows:

$$k_0 \qquad k_2 \qquad k_4 \qquad k_6 \qquad \ldots$$

$$k_1 \qquad k_3 \qquad k_5 \qquad k_7 \qquad \ldots$$

$$k_2 - \frac{k_0}{k_1} k_3 \qquad k_4 - \frac{k_0}{k_1} k_5 \qquad k_6 - \frac{k_0}{k_1} k_7 \qquad \ldots$$

This process is continued in the same manner until there is only one term. The system is then stable if, and only if, the first term of each row is greater than zero. The computational advantage of this scheme is that instability is indicated immediately by the appearance of a negative term (not necessarily the first of the row) and the computations can then be stopped.

BOD FORCING FUNCTION

Individual models can be constructed for each of the inputs in Eq. 26 (Thomann[8]) if a complete description of the independent effects of each of the inputs on DO is desired. However, the BOD input function is of primary interest to sanitary engineers and it is of value to derive relationships between the waste effluents and the stream BOD and between the stream BOD and the DO. There is really little necessity to concentrate on stream BOD alone because in practically all cases, it is not considered as a water quality variable. It only results as an output due to discharges from treatment plants and then immediately begins to act as an input in the dissolved oxygen representation. If a valid description can be obtained directly between the effluent BOD and the stream DO, the field measurements of stream BOD may be taken only in the context of verifying a theoretical approach. The following development will indicate this direct relationship using a systems analysis approach similar to that described previously.

As is seen from Eq. 26, the BOD forcing function is of the form $d_1(t)\ L_1(t)$ in which $d_1(t)$ is the decay coefficient and $L_1(t)$ is the quantity of organic matter present in terms of its ultimate use of oxygen. It may be postulated that this organic sink of oxygen is a function primarily of (a) water temperature, acting on the decay coefficient; (b) advection and diffusion of L(t); and (c) direct sources of wastes such as discharges from municipal and industrisl waste treatment plants. The product term can be written in terms of a mean steady state value and the deviation from the mean, i.e.,

$$d(t)\ L(t) = (\overline{d} + d'(t))\ (\overline{L} + L'(t))$$

$$= \overline{dL} + \overline{d}L'(t) + \overline{L}d'(t) + d'(t)\ L'(t) \quad \ldots\ldots\ldots\ldots (84)$$

The first term contributes only to the mean value DO response and is not of interest. The second term represents the variation in the utilization of oxygen due solely to changes in the BOD while the decay coefficient is held at its mean value. The third term indicates the effect of varying decay coefficient (due primarily to varying temperature) with the BOD at its mean value. The fourth term, a second order interaction effect, represents the variation of the

organic sink as both the decay coefficient and BOD vary simultaneously. Eq. 84 then shows that in a time varying system, the BOD input can be written as a sum of four sub-inputs each of which assumes a particular role.

When the variation of L'(t) is to be described, an analysis similar to the DO system presentation can be performed. This leads to a set of equations parallel to Eqs. 34:

$$\left.\begin{aligned} a_{11}(p)L_1 + a_{12}L_2 + a_{13}L_3 + \ldots + a_{1n}L_n &= J_1(t) \\ a_{21}L_1 + a_{22}(p)L_2 + a_{23}L_3 + \ldots + a_{2n}L_n &= J_2(t) \\ \cdot \quad\quad \cdot \quad\quad \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \quad\quad \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \quad\quad \cdot \quad\quad \cdot \\ a_{n1}L_1 + a_{n2}L_2 + \ldots + a_{nn}(p)L_n &= J_n(t) \end{aligned}\right\} \quad \ldots . (85)$$

in which all terms have been previously defined except $J_k(t)$ which is the rate of discharge of BOD and is computed from $L_{pk}Q_{pk}/V_k$ in which L_{pk} is the BOD of the plant effluent, Q_{pk} is the effluent flow, and V_k represents the volume of segment k. Of course, there are interactions similar to those described previously between the BOD of the effluent and the flow. For purposes of this analysis, however, the input treatment plant discharge will be left in its general form, $J_k(t)$.

In Eq. 85, the a_{ij} terms are functions of the flow between sections, the exchange coefficient, and the mean decay coefficient whereas in Eq. 34, the reaeration phenomenon replaced the decay mechanism. Now, if an analysis is performed parallel to that indicated for the solution of $c_j(t)$, the BOD response is given by

$$L'_i(t) = J_{ok}\left[A_{ki}(\omega)\right]_{JL} \cos\left\{\omega t + \left[\theta_{ki}(\omega)\right]_{JL}\right\} \quad \ldots . (86)$$

in which $L_i(t)$ is the response in system i due to a harmonic forcing function of amplitude J_{ok} applied in system k. (A zero phase angle of the input has been assumed for simplicity.) The subscript JL on the transfer function indicates the treatment plant BOD to stream BOD system. Eq. 86 thus indicates the variation of stream BOD in section i resulting from a discharge into section k. The total BOD response in i would be given by a summation over all k sections into which plant effluents discharge; however, only one discharge will be treated herein. The inclusion of more than one discharge merely requires the linear superposition mentioned previously.

When Eq. 86 is multiplied by the mean decay coefficient, it then may act as an input forcing function for the DO system. Following Eq. 42, the BOD input can be written as

$$f_i(t) = \bar{d}L'_i(t) = \bar{d}\left[\frac{L_{oi}}{2}e^{i\omega t} + \frac{L_{oi}}{2}e^{-i\omega t}\right] \quad \ldots . (87)$$

in which

$$L_{oi} = J_{ok} \left[A_{ki}(\omega)\right]_{JL} e^{i\left[\theta_{ki}(\omega)\right]_{JL}} \quad \ldots\ldots\ldots\ldots (88)$$

The substitution of Eq. 87 into Eq. 43 and formally continuing to Eq. 56 provides the final solution for $c_j(t)$. Hence,

$$c_j(t) = \bar{d} \sum_{i=1}^{n} J_{ok} \left[A_{ki}(\omega)\right]_{JL} \left[A_{ij}(\omega)\right]_{Lc} \cos \left[\omega t + \left[\theta_{ki}(\omega)\right]_{JL} + \left[\theta_{ij}(\omega)\right]_{Lc}\right] \quad \ldots\ldots\ldots\ldots\ldots (89)$$

in which the subscript Lc denotes the stream BOD to DO system. The summation over all i sections is necessary because a single discharge will theoretically lead to a dispersion of BOD into all systems of the body of water. As indicated previously, it is always interesting to describe the cause and effect relationship between a discharge and the resulting DO in the stream. Eq. 89 provides an explicit relationship for this description.

It can be seen that a double attenuation of the plant effluent amplitude occurs and also that the resulting DO phase angle is composed of two shifts, one from the plant BOD to stream BOD system and the second from the stream BOD to DO system. Now, in general, the amplitude attenuation factor $A(\omega)$ decreases with increasing frequency. Hence, high frequency (short period) fluctuations in $J_k(t)$ will tend to be damped by the double attenuation and will have little effect on the stream DO.

The second BOD sub-input, $\bar{L}\, d'(t)$, which expresses the variation in organic matter due to changes in the decay coefficient, can be described by considering $d'(t)$ to be primarily a function of the water temperature. It has traditionally been assumed that temperature affects the decay coefficient according to the Vant-Hoff-Arhennius relationship for chemical kinetics. This relationship leads to an equation of the form

$$\frac{d_T}{d_{20}} = k^{T-20} \quad \ldots\ldots\ldots\ldots\ldots\ldots (90)$$

in which d_T and d_{20} are the decay coefficients at temperature T(°C) and 20°C, respectively, and k is a factor varying from 1.02 to 1.05. When a Cartesian coordinate system is used, the degree of curvature of the relationship depends on the value of k. The expression could also be represented by a second-order polynomial. However, the use of data obtained by H. B. Gotaas[10] indicates that a straight line relationship is appropriate for most purposes. It should be noted that the data given were obtained from an analysis of the decay coefficients of various types of sewage effluents and not for the decay mechanism

[10] "Effect of Temperature in Biochemical Oxidation of Sewage," by H. B. Gotaas, Sewage Works Journal, Vol. 20, No. 3, 1948, pp. 441-447.

in a body of water. However, it is believed that the general trend will not differ significantly from what would be expected in a stream. It should also be noted that the following analysis is not restricted to a straight line function but can be expanded to include any order of polynomial or other functional forms (Thomann[8]). The linear relationship is used here to illustrate a procedure, and in this particular input it appears to be the most suitable expression.

If

$$d(t) = a + b\ T(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (91)$$

in which T is temperature, in °C, and

$$T(t) = \overline{T} + T'(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (92)$$

in which

$$T'(t) = T_A \cos\left(\omega t + \theta_T\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (93)$$

Then,

$$d(t) = a + b\,\overline{T} + b\ T_A \cos\left(\omega t + \theta_T\right) = \overline{d} + b\ T_A \cos\left(\omega t + \theta_T\right) \quad \ldots (94)$$

Again, the single harmonic representation of T'(t) can be generalized to any function of time. The deviation of the decay coefficient from its mean value is then simply

$$d'(t) = b\ T_A \cos\left(\omega t + \theta_T\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (95)$$

The linear relationship between d and T thus leads to a particular case of the transfer function, as shown by Eq. 95. In Eq. 95, the transfer function is a constant, i.e., the slope b and the phase angle shift is of course equal to zero. If estimates of the amplitude of temperature, T_A, and the phase angle, θ_T, are at hand. Eq. 95 provides a ready means for estimating the variation of d'(t).

The third sub-input given by the second order interaction d'(t) L'(t) is best handled by a product analysis using Eqs. 86 and 95. Hence,

$$d'L' = \left[b\ T_A \cos\left(\omega t + \theta_T\right)\right]\left[J_0 A_{JL} \cos\left(\omega t + \theta_{JL}\right)\right]$$

$$= D \cos\left(2\ \omega\ t + \theta_{JL} + \theta_T\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (96)$$

in which

$$D = J_0 A_{JL}\ b\ T_A \quad \ldots\ldots\ldots\ldots\ldots\ldots (97)$$

and the constant term cos (θ_{JL} - θT) has been absorbed into the mean value $\overline{dL}$. Eq. 96 indicates that the higher order interaction term between the varying decay coefficient and the varying stream BOD results in a forcing function of twice the frequency of the original input, in this case, J(t) and T(t). If, there-

FIG. 3.—BLOCK DIAGRAM REPRESENTATION OF BOD FORCING FUNCTION

fore, the treatment plant effluent is almost periodic with a period of, for example, 365 days or $2\pi/365$ radians, then the interaction term will be periodic with a period of 182.5 days or $2(2\pi/365)$ radians. Because the DO system is linear, the oxygen output due to the interaction input will also vary with twice the basic frequency of the treatment plant effluent. The analysis thus illustrates why certain periodicities are noted in an observed DO record. Although there may be sufficient evidence for the existence of an annual variation in DO, the reasons for a semiannual periodicity are not immediately apparent. In this case, this latter effect is due to the interaction terms in the inputs. In general, new output frequencies can be produced when nonlinearities are present (as previously) or when the system parameters are time varying. Frequencies for which a physical explanation is not immediately available will occur in the final output DO record and, following the preceding analyses, may be entirely real and deserving of a cause and effect analysis.

A simplified block diagram representation of the BOD forcing function is shown in Fig. 3. As indicated, the water temperature and treatment plant systems are not considered in detail. The outputs from these systems assume input characteristics in the decay and stream BOD systems. The new outputs are then combined in a multiplying system, the final output of which enters the DO system.

Fig. 3 also illustrates how the systems analysis can be continued "backwards," for example, to the treatment plant itself. An input-output analysis of the plant using the characteristics of the influent and plant units would provide a means for computing the expected effluent level and hence the stream BOD and final DO. However, for many purposes, the analysis may cease when a variable has been reached that is more easily described or forecasted than the variable of interest. There is little need to describe the water temperature system in detail because the output is reasonably well-known in most situations and is certainly known to a degree of several orders of magnitude higher than the DO. This then would form a logical starting point for describing the DO in a stream provided of course that water temperature is not the variable of interest.

SUMMARY AND CONCLUSIONS

The description of dissolved oxygen in a body of water is adaptive to a mathematical analysis using the techniques of systems analysis. The model as presented in its general form embodies all types of inputs and is capable of assessing the effects of any one time varying input in one section of the water body on the DO in any other section. Furthermore, there are no analytical difficulties in viewing the model from a general water quality point of view where the dependent variable can assume the role of any water quality constituent. In this most general sense, different types of forcing functions must be considered and the system parameters may change somewhat. Hence, if the model is applied to salinity intrusion in an estuary, the only forcing function of any significance would be the salt influx from the ocean. Decay mechanisms can be assumed to be absent and the transfer function then depends only on the net flow through the estuary and the magnitude of the eddy exchange phenomenon.

From a computational viewpoint, the model is capable of being solved in several different ways. Of course, digital computation can be used to obtain the transfer function, analyze input forcing functions, and generate a computed DO output (Thomann[8]). Because the model is basically founded on network theory, analog computation appears to be particularly appropriate. This can be done either through use of a general purpose analog or through use of custom-built analogs particular to a specific river or estuary. The analyses as presented herein require some type of automatic computation, especially for the construction of the transfer function. However, once good estimates of this function have been obtained for a particular river, the model is of value for design purposes even if a coarser analysis is used. The major part of the DO phenomenon can usually be accounted for by a relatively few input frequencies provided the transfer characteristics are available.

Finally, because sanitary engineers are concerned primarily with the management of the water system, the model provides a description that is particularly amenable to the application of control engineering theory. In this case, a desired level of the output is given and the problem then is to determine either (a) the level of input needed, (b) the changes in the transfer function that may be necessary or (c) a suitable combination of both (a) and (b). In all situations, the cost of imposing the required changes can be evaluated and the problem may be solved in an optimum sense. M. J. Sobel[11] has presented an analysis of the model in a programming problem and has indicated how the transfer function and its transform, the impulsive response function, are used in determining the optimum utilization of funds for waste treatment and water quality control.

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11 "On the Management of the Quality of Natural Water Systems," by M. J. Sobel, paper M-1, presented at the Annual Meeting of the Operations Research Society of America, Cleveland, Ohio, May, 1963.

APPENDIX.—NOTATION

The following symbols have been adopted for use in this paper:

$A_{ij}(\omega)$ = amplitude attenuation of $f_i(t)$ for solution of $c_j(t)$;

a_{ij} = element in i^{th} row and j^{th} column of determinant D;

$a_k(k=0,1,\ldots n)$ = constant coefficient in differential equation for c_j (Eq. 35);

$b_r(r=0,1,\ldots m:m<n)$ = constant coefficient in right side of Eq. 35;

$c_j(t)$ = dissolved oxygen concentration (DO) in system j;

$c_{si}(t)$ = saturation value of DO in system i;

D(p) = determinant;

$d_i(t)$ = decay coefficient in system i;

$E_{k'k}$ = exchange coefficient between systems k' and k;

$F(i\omega)$ = Fourier transform of $f_i(t)$;

$f_i(t)$ = total forcing function in system i;

i = $\sqrt{-1}$, (also system in which f_i is applied);

$J_i(t)$ = treatment plant forcing function in system i;

j = system for which solution of c_j is desired;

K = diffusion coefficient;

k = index usually 1, 2, . . ., n;

k(t) = over-all impulsive response function;

$L_i(t)$ = organic sink (biochemical oxygen demand) in system i;

M(p) = minor of determinant D;

n = integer equal to number of systems;

$P_{ij}(\omega)$ = real part of transfer function for $c_j(t)$ and $f_i(t)$;

p = operator = d/dt;

$Q_{ij}(\omega)$ = imaginary part of transfer function for $c_j(t)$ and $f_i(t)$;

$Q_{k'k}$ = flow between systems k' and k;

r_i = reaeration coefficient in system i;

$T(t)$ = water temperature;

T = period of one cycle;

t = time;

V_i = volume of system i;

$\alpha, \beta, \gamma, \delta$ = polynomials in transfer function (Eqs. 45);

$\delta(t-t_o)$ = delta function = $\int_{-\infty}^{\infty} \delta(t - t_o)\, dt = 1$;

$\theta_{ij}(\omega)$ = phase angle shift for $c_j(t)$ due to $f_i(t)$;

λ^n = roots of characteristic Eq. 83;

$\xi_{k'k}$ = advection factor between systems k' and k;

π = 3.14159;

$\phi_{ij}(i\omega)$ = transfer function for $f_i(t)$ and $c_j(t)$; and

ω = angular frequency.

KEY WORDS: computers; estuaries; mathematics; models; sanitary engineering; streams

ABSTRACT: Systems analysis techniques are used to construct a mathematical model of the variation of dissolved oxygen (DO) in a body of water. The application of these techniques leads to a set of ordinary differential equations that can be solved for the DO response in one section due to an input in any other section. The function that expresses this input-output relationship is shown to be dependent on the hydrodynamic nature of the stream or estuary. Successive use of a linear systems analysis leads to an analytical relationship between the effluent BOD and the resulting DO in the stream.

REFERENCE: "Mathematical Model for Dissolved Oxygen," by Robert V. Thomann, Journal of the Sanitary Engineering Division, ASCE, Vol. 89, No. SA5, Proc. Paper 3680, October, 1963, pp. 1-30.